Journal of Differential Equations **160**, 263–282 (2000)
Article ID jdeq.1999.3673, available online at http://www.idealibrary.com on IDEAL®

A Decomposition Theorem for Bounded Solutions and the Existence of Periodic Solutions of Periodic Differential Equations

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Received August 24, 1998; revised February 10, 1999

DEDICATED TO THE MEMORY OF PROFESSOR T. YOSHIZAWA

We prove a decomposition theorem for bounded uniformly continuous mild solutions to τ-periodic evolution equations of the form $dx/dt = A(t)\,x + f(t)$ $(*)$ with (in general, unbounded) τ-periodic $A(\cdot)$, τ-periodic $f(\cdot)$, and compact monodromy operator. By this theorem, every bounded uniformly continuous mild solution to $(*)$ is a sum of a τ-periodic solution to $(*)$ and a quasi periodic solution to its homogeneous equation. An analog of this for bounded solutions has been proved for abstract functional differential equations $dx/dt = Ax + F(t)\,x_t + f(t)$ with finite delay, where A generates a compact semigroup. As an immediate consequence, the existence of such a solution implies the existence of a τ-periodic solution to the inhomogeneous equation as well as a formula for its Fourier coefficients. This, even for the classical case of equations, improves considerably the previous results on the subject.

Key Words: periodic evolution equation; abstract functional differential equation; compact monodromy operator; spectrum of bounded function; decomposition theorem; periodic solution; Fourier coefficients.

0022-0396/00 $35.00

1. INTRODUCTION AND PRELIMINARIES

Let us consider τ-periodic evolution equations of the form

$$\frac{dx}{dt} = A(t)\,x + f(t), \qquad x \in \mathbf{X},\ t \in \mathbf{R}, \tag{1}$$

$$\frac{dx}{dt} = A(t)\,x, \qquad x \in \mathbf{X},\ t \in \mathbf{R}, \tag{2}$$

where $A(\cdot)$ is a τ-periodic (unbounded) operator on a Banach space $\mathbf{X}$ and f is an $\mathbf{X}$-valued τ-periodic continuous function.

Assume that Eq. (2) is well posed, that is, there exists a τ-periodic evolutionary process $(U(t,s))_{t \geqslant s}$ which satisfies, among other things, the conditions of Definition 1 below. Then the asymptotic behavior of solutions to Eq. (1), such as stability and (almost) periodicity, is of particular interest, which makes them the subject for numerous researches.

A classical result of the theory of ordinary differential equations says that *if* $\dim \mathbf{X} < \infty$ *and there exists a bounded solution to Eq.* (1), *then there exists a τ-periodic solution to Eq.* (1) (for the proof see, e.g., [A, Theorem 20.3]). This serves as a starting point for many interesting researches. Namely, Chow and Hale proved the analog of this for functional differential equations by using their own results on fixed points of affine maps [C–H]; Daners and Medina [D–M, Theorem 6.11] extended the method used in [A, Theorem 20.3] to prove the analog for parabolic equations. Recently Shin and Naito [S–N] have extended Chow and Hale's method to abstract functional differential equations to prove the analog of the above result for abstract functional differential equations with infinite delay. In this direction see also the recent papers [L–L–L, Mu1, H–M–Y]. We refer the reader to [H–V, H–K, La, Li, N–M, Pr, S–N, Ya, Yo] and the references therein for results on the existence and uniqueness of τ-periodic solutions to Eq. (1). As there are too large a number of papers dealing with this subject with diverse methods we cannot add all of them to our list of references. We hope that the reader can start from the references above to get more information on the subject.

Though various methods have been used to show the analog of the above result for various classes of equations, to the best of our knowledge, there has been no method so far which gives an exact description of the bounded solutions and the method for computation of the τ-periodic solutions under question.

It is the purpose of this paper to make an attempt to fill this gap. In fact, we will prove a decomposition theorem for bounded uniformly continuous mild solutions to Eq. (1) which says that every bounded uniformly continuous mild solution to Eq. (1) is a sum of a τ-periodic mild solution to Eq. (1) and a quasi periodic mild solution to the corresponding homogeneous equation (2). As an immediate consequence, it follows that the existence of a bounded uniformly continuous mild solution yields the existence of a τ-periodic mild solution. Moreover, by the decomposition theorem a formula for the Fourier coefficients of the τ-periodic solution is given. We should emphasize that in most of the cases frequently met in applications the uniform continuity follows readily from the boundedness of mild solutions (see Propositions 2 and 3 and examples for parabolic equations in Sections 2, 3, and 4 for more information). To our knowledge the decomposition theorem which is proved in this paper is new in its type, even for ordinary differential equations; it may have further applications to the asymptotic behavior of solutions to evolution equations, which will be discussed in our future investigation.

We now give an outline of our paper. Section 1 gives a short introduction to the matter studied in the paper and some basic properties of the spectral theory of functions. In Section 2 the decomposition theorem and its corollary are presented. Proposition 2 shows that the uniform continuity follows readily from boundedness in many cases frequently met in applications. Section 3 is devoted to abstract functional differential equations of the form

$$\frac{dx}{dt} = Ax + F(t)\, x_t + f(t),$$

where A generates a compact C_0-semigroup, $F(\cdot)$ is τ-periodic in t, continuous, and linear and f is τ-periodic, $x_t(\theta) = x(t+\theta)$, $\theta \in [-r, 0]$ for a given positive r. In this case, uniform continuity follows from boundedness (Proposition 3) and all results from Section 2 can be extended to this case (Theorem 2 and Corollary 2). In Section 4 we consider parabolic equations. For this kind of equations, uniform continuity follows from the boundedness and mild solutions are classical ones.

Throughout the paper we will use the following notations: $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$, $\mathbf{C}$ stand for the sets of natural numbers, integers, real numbers, and complex numbers, respectively, and $\mathbf{X}$ will denote a given complex Banach space. If T is a linear operator on $\mathbf{X}$, then $D(T)$ stands for its domain. Given two Banach spaces $\mathbf{X}$ and $\mathbf{Y}$, by $L(\mathbf{X}, \mathbf{Y})$ we will denote the space of all bounded linear operators from $\mathbf{X}$ to $\mathbf{Y}$. As usual, $\sigma(T)$, $\rho(T)$, and $R(\lambda, T)$ are the notations for the spectrum, resolvent set, and resolvent of the operator T. The notations $\mathrm{BC}(\mathbf{R}, \mathbf{X})$, $\mathrm{BUC}(\mathbf{R}, \mathbf{X})$, $\mathrm{AP}(\mathbf{X})$ will stand for the spaces of all $\mathbf{X}$-valued bounded and bounded uniformly continuous functions on $\mathbf{R}$ and their subspace of almost periodic (in Bohr's sense) functions, respectively.

In the paper we will use the notion of *Carleman spectrum* of a bounded continuous function u on the whole line, denoted by $\mathrm{sp}(u)$, consisting of $\xi \in \mathbf{R}$ such that the Fourrier–Carleman transform of u,

$$\hat{u}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t} u(t)\, dt, & (\mathrm{Re}\ \lambda > 0); \\ -\int_0^\infty e^{\lambda t} u(-t)\, dt, & (\mathrm{Re}\ \lambda < 0), \end{cases} \tag{3}$$

has no holomorphic extension to a neighborhood of $i\xi$ (see, e.g., [Pr, pp. 19–27]). Basic properties of the spectrum of a function and their relation to the behavior of the function are listed below for the reader's convenience.

PROPOSITION 1. *Let* $f, g_n \in \mathrm{BUC}(\mathbf{R}, \mathbf{X})$ *such that* $\lim_{n \to \infty} \|g_n - f\| = 0$. *Then*

(i) $\mathrm{sp}(f)$ *is closed*,

(ii) $\mathrm{sp}(f(\cdot + h)) = \mathrm{sp}(f)$,

(iii) *If* $\alpha \in \mathbf{C} \setminus \{0\}$, *then* $\mathrm{sp}(\alpha f) = \mathrm{sp}(f)$,

(iv) *If* $\mathrm{sp}(g_n) \subset \Lambda$ *for all* $n \in \mathbf{N}$, *then* $\mathrm{sp}(f) \subset \bar{\Lambda}$,

(v) $\mathrm{sp}(\psi + f) \subset \mathrm{sp}(f) \cup \mathrm{sp}(\psi)$, $\forall \psi \in \mathrm{BC}(\mathbf{R}, \mathbf{X})$,

(vi) *If* u *is uniformly continuous*, $\mathrm{sp}(u)$ *is countable, and* $\mathbf{X}$ *does not contains any subspace which is isomorphic to the space of sequences* c_0, *then* u *is almost periodic*,

(vii) *If* u *is uniformly continuous and*, $\mathrm{sp}(u)$ *is discrete, then* u *is almost periodic*.

Proof. For the proof we refer the reader to [Pr, Proposition 0.4, p. 20, Theorem 0.8, p. 21, [A–S; L–Z, Chap. 6]. ∎

If u is an almost periodic function, then $\mathrm{sp}(u)$ is the closure of the Bohr spectrum of u, i.e., the set of all real numbers λ such that the following limit exists:

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda\xi} u(\xi)\, d\xi.$$

As is known, for an almost periodic function u, there are at most countably such real numbers λ (see, e.g., [A–P, L–Z]).

2. PERIODIC EVOLUTION EQUATIONS

In this section we will deal with mild solutions to the inhomogeneous integral linear equations which arise from Eq. (1). The following notation will be used throughout the paper:

DEFINITION 1. A family of bounded linear operators $(U(t, s))_{t \geqslant s}$ $(t, s \in \mathbf{R})$ from a Banach space $\mathbf{X}$ to itself is called a *τ-periodic strongly continuous evolutionary process* if the following conditions are satisfied:

(i) $U(t, t) = I$ for all $t \in \mathbf{R}$,

(ii) $U(t, s)\, U(s, r) = U(t, r)$ for all $t \geqslant s \geqslant r$,

(iii) The map $(t, s) \mapsto U(t, s)\, x$ is continuous for every fixed $x \in \mathbf{X}$,

(iv) $U(t+\tau, s+\tau) = U(t, s)$ for all $t \geqslant s$,

(v) $\| U(t, s) \| < N e^{\omega(t-s)}$ for some positive N, ω independent of $t \geqslant s$.

We say that u is a *mild solution* to Eq. (1) on $\mathbf{R}$ if and only if

$$u(t) = U(t, s)\, u(s) + \int_s^t U(t, \xi)\, f(\xi)\, d\xi, \qquad \forall t \geqslant s. \tag{4}$$

As a standing hypothesis in this section we assume that the monodromy operator which, by definition, is $P := U(\tau, 0)$ is compact. In the sequel we call λ_k, $k = 1, ..., N$, *basic oscillatory exponents* of Eq. (2) if $0 < \lambda_1 < \cdots < \lambda_N < 2\pi/\tau$ and $\{e^{i\lambda_k \tau}, k = 1, ..., N\} = [\sigma(P) \backslash \{1\}] \cap S^1$, where S^1 is the unit circle, i.e., $S^1 := \{e^{i\eta}, \eta \in \mathbf{R}\}$. In this section we will use the abbreviation $U(t) = U(t+\tau, t)$, $\forall t \in \mathbf{R}$. From time to time we may also call $U(t)$ a monodromy operator by abuse of terminology and especially by the fact that $\sigma(P) \backslash \{0\} = \sigma(U(t)) \backslash \{0\}$, $\forall t \in \mathbf{R}$ (see [He, Lemma 7.2.2, p. 197]).

2.1. *Formulation of the Decomposition Theorem and Its Consequences*

This section is aimed at proving the following theorem which gives an exact description of bounded uniformly continuous mild solutions to Eq. (1) modulo a τ-periodic function:

THEOREM 1. *Let $u(\cdot)$ be a bounded uniformly continuous mild solution to Eq. (1). Then it is of the form*

$$u(t) = u_0(t) + \sum_{k=1}^{N} e^{i\lambda_k t} u_k(t), \tag{5}$$

where u_0 *is a bounded uniformly continuous mild* τ*-periodic solution to the inhomogeneous equation* (1), $u_k, k=1, ..., N$, *are* τ*-periodic solutions to Eq.* (1) *with* $f = -i\lambda_k u_k$, *respectively,* $v(t)=\sum_{k=1}^{N} e^{i\lambda_k t} u_k(t)$ *is a quasi periodic mild solution to the homogeneous equation* (2), *and* $\lambda_1, ..., \lambda_N$ *are the basic oscillatory exponents of Eq.* (2).

From Theorem 1 in particular follows

COROLLARY 1. *Equation* (1) *has a bounded uniformly continuous* τ*-periodic mild solution if and only if it has a bounded uniformly continuous mild solution. Moreover, if* u *is a bounded uniformly continuous mild solution to Eq.* (1), *then the two-sided sequence*

$$a_k = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-(2ik\pi/\tau)\,\xi} u(\xi)\, d\xi, \; k \in \mathbf{Z} \tag{6}$$

determines the Fourier coefficients of a τ*-periodic mild solution to Eq.* (1).

Proof. The corollary is an immediate consequence of Theorem 1 and the general theory of almost periodic functions (see, e.g., [A–P, L–Z]). ▮

The first assertion of Corollary 1 is a generalization of a well-known result from ordinary differential equations (see [A, Theorem 20.3]). This result has been generalized to functional differential equations in [C–H] and to parabolic equations in [D–M, Theorem 6.11, p. 81] by the Fredholm operator method. Recently, it was generalized to abstract functional differential equations in [S–N] with infinite delay. In the next section we will discuss the analog of Theorem 1 for abstract functional differential equations with finite delay.

2.2. *Proof of the Decomposition Theorem*

The proof of Theorem 1 will be based on the following lemmas:

LEMMA 1. *Let* $w_\eta \in \mathrm{BUC}(\mathbf{R}, \mathbf{X})$, $\forall \eta \in \mathbf{Y}$, *where* $\mathbf{Y}$ *is a* (*metric*) *subspace of* $\mathbf{R}$, *such that*

$$w_\eta(t) = \sum_{j=0}^{m} e^{i\lambda_j t} w_{\eta,j}(t), \tag{7}$$

where $0 \leqslant \lambda_0 < \lambda_1 < \cdots < \lambda_m < 2\pi/\tau$ *and* $w_{\eta,j}$ *are* τ*-periodic continuous functions. Moreover, let* $\lim_{\eta \to \eta_0} w_\eta = 0$ *in* $\mathrm{BUC}(\mathbf{R}, \mathbf{X})$. *Then for every* $j=0, 1, ..., m$, $\lim_{\eta \to \eta_0} w_{\eta,j} = 0$ *in* $\mathrm{BUC}(\mathbf{R}, \mathbf{X})$.

Proof. Without loss of generality we can assume that $\lambda_0 = 0$. Now we are going to prove the lemma by induction. If $m = 0$, then the lemma is obvious. Suppose that the lemma holds true for $m = p-1$. Now set

$$
\begin{aligned}
v_\eta(t) &:= w_\eta(t) - w_\eta(t-\tau) \\
&= \left[w_{\eta,0}(t) + \sum_{k=1}^{p} e^{i\lambda_k t} w_{\eta,k}(t) \right] - \left[w_{\eta,0}(t-\tau) + \sum_{k=1}^{p} e^{i\lambda_k (t-\tau)} w_{\eta,k}(t-\tau) \right].
\end{aligned}
$$

Since $w_{\eta,j}(t)$ is τ-periodic, we have

$$
\begin{aligned}
v_\eta(t) &= \sum_{k=1}^{p} e^{i\lambda_k t} (1 - e^{-i\tau\lambda_k})\, w_{\eta,k}(t) \\
&= \sum_{k=1}^{p} e^{i\lambda_k t} v_{\eta,k}(t), \qquad (8)
\end{aligned}
$$

where $v_{\eta,k}(t) := (1 - e^{-i\tau\lambda_k})\, w_{\eta,k}(t)$. By the induction assumption from this expression it follows that $v_{\eta,k}(t)$, $k = 1, ..., p$, converge to zero uniformly in t. So do $w_{\eta,k}(t)$, $k = 1, ..., p$, and $w_{\eta,0}(t)$. ∎

We now recall that for periodic evolutionary processes with compact monodromy operator there always exist partial Floquet representations. This was first discussed in [Ha, Chap. 8] for functional differential equations, and then in [He, pp. 190–198] (see also [D–M, Chap. 2]) for parabolic equations. For the reader's convenience we recall here the main idea of this representation which will play a particular role in our construction. Let us denote

$$
\sigma_1 = \{\lambda \in \sigma(P) : |\lambda| \geqslant 1\}.
$$

Thus, σ_1 is finite. As is known, there exists a partial Floquet representation for the evolutionary process $(U(t,s))_{t \geqslant s}$ (see, e.g., [He, pp. 198–200]). We now make it more clear. Put

$$
E_1(t) = \int_\gamma R(\lambda, U(t))\, d\lambda,
$$

where γ is a contour disjoint from $\sigma(P(t))$ (which, excluding the point 0, is independent of t) enclosing σ_1 and excluding $\sigma_2 := \sigma(P) \backslash \sigma_1$. Then $E_1(t)$ is a projection. Setting $\operatorname{Im} E_1(t) := X_1(t)$ we have $U(s_0)|_{X_1(s_0)} = e^{\tau C}$, where

$$
C := \frac{1}{2\pi i \tau} \int_\gamma (\lambda - U(s_0)|_{X_1(s_0)})^{-1} \log \lambda\, d\lambda.
$$

Also

$$E_1(t)\, U(t,s) = U(t,s)\, E_1(s), \qquad \forall t \geqslant s.$$

Define for $t \geqslant s_0$

$$P(t) := U(t, s_0)|_{X_1(s_0)} e^{-C(t-s_0)}$$

Then $P(t+\tau) = P(t)$. Moreover, setting $u_1(t) := E_1(t)\, u(t)$, $u_2(t) = E_2(t)\, u(t)$, where $E_2(t) = I - E_1(t)$ and $u(t)$ is a mild solution we have the equation

$$\begin{aligned} u_1(t) &= P(t)\, e^{\tau C} P^{-1}(t-\tau)\, u_1(t-\tau) + E_1(t) \int_{t-\tau}^{t} U(t,\xi)\, f(\xi)\, d\xi, \\ u_2(t) &= E_2(t)\, U(t,s)\, u_2(s) + E_2(t) \int_{s}^{t} U(t,\xi)\, f(\xi)\, d\xi, \qquad \forall t \geqslant s. \end{aligned} \tag{9}$$

Moreover, there are positive constants M, ω such that

$$\|E_2(t)\, U(t,s)\, E_2(s)\| \leqslant Me^{-(t-s)\,\omega}, \qquad \forall t \geqslant s. \tag{10}$$

Hence by the transformation

$$Q(t) = P^{-1}(t)\, E_1(t) + E_2(t) \tag{11}$$

with the property that $(t, x) \mapsto Q(t)\, x$ is continuous, and τ-periodic in t and $\sup_t \|Q(t)\| < \infty$, from the above system of equations we have

$$\begin{aligned} v_1(t) &= Bv_1(t-\tau) + F_1(t), \\ v_2(t) &= B(t)\, v_2(t-\tau) + F_2(t), \end{aligned} \tag{12}$$

where $B = e^{\tau C}$, $B(t) = E_2(t)\, U(t, t-\tau)\, E_2(t-\tau)$, $v(t) = Q(t)\, u(t) = v_1(t) + v_2(t)$, $v_1(t) \in X_1(t)$, $v_2(t) \in X_2(t) = \mathrm{Im}(I - E_1(t))$. Below we will seperately deal with the equations in the system (12).

LEMMA 2. *Assume that $B(t)$ satisfies the following conditions: $(t, x) \mapsto B(t)\, x$ is continuous, $B(t+\tau) = B(t)\ \forall t$, and $\sup_t \|B(t)\| < 1$. Then for every τ-periodic continuous function g the functional equation*

$$x(t) = B(t)\, x(t-\omega) + g(t), \qquad \forall t \in \mathbf{R}, \tag{13}$$

where ω is some fixed real number, has a unique bounded continuous solution. Moreover, this solution is τ-periodic.

Proof. Define the operator

$$Tv(t) := B(t)\, v(t-\omega), \qquad v \in BC(\mathbf{R}, \mathbf{X}), \quad \forall t, \tag{14}$$

where ω is the fixed real number. Then T is a well-defined operator in $BC(\mathbf{R}, \mathbf{X})$. The operator $Gv := Tv + g$, $\forall v \in BC(\mathbf{R}, \mathbf{X})$ is a strict contraction in $BC(\mathbf{R}, \mathbf{X})$. Hence, it has a unique fixed point v_0 which solves Eq. (13). Moreover,

$$v_0 = \lim_{n \to \infty} G^n g.$$

As for every n $G^n g$ is τ-periodic, so is v_0. ∎

LEMMA 3. *Let f be τ-periodic and continuous, and let u be a bounded uniformly continuous solution to the functional equation*

$$u(t) = Bu(t - \tau) + f(t), \qquad \forall t. \tag{15}$$

Then

$$e^{i\tau\, \mathrm{sp}(u)} \subset \{1\} \cup (\sigma(B) \cap S^1). \tag{16}$$

Proof. Taking the Carleman transform of u we have

$$\hat{u}(\lambda) = \begin{cases} \displaystyle\int_0^\infty e^{-\lambda t} u(t)\, dt, & (\mathrm{Re}\,\lambda > 0); \\ \displaystyle -\int_0^\infty e^{\lambda t} u(-t)\, dt, & (\mathrm{Re}\,\lambda < 0). \end{cases} \tag{17}$$

Since $u(t) = Bu(t - \tau) + f(t)$, we have

$$\hat{u}(\lambda) = \begin{cases} \displaystyle\int_0^\infty e^{-\lambda t} Bu(t - \tau)\, dt, & (\mathrm{Re}\,\lambda > 0); \\ \displaystyle -\int_0^\infty e^{\lambda t} Bu(-t - \tau)\, dt, & (\mathrm{Re}\,\lambda < 0) \end{cases} + \hat{f}(\lambda)$$

$$= \begin{cases} \displaystyle e^{-\lambda\tau} B \int_{-\tau}^\infty e^{-\lambda t} u(t)\, dt, & (\mathrm{Re}\,\lambda > 0); \\ \displaystyle -e^{-\lambda\tau} B \int_\tau^\infty e^{\lambda t} u(-t)\, dt, & (\mathrm{Re}\,\lambda < 0) \end{cases} + \hat{f}(\lambda)$$

$$= e^{-\lambda\tau} B\hat{u}(\lambda) + \phi(\lambda) + \hat{f}(\lambda),$$

where $\phi(\lambda) := e^{-\lambda\tau} B \int_{-\tau}^0 e^{-\lambda\xi} u(\xi)\, d\xi$. It may be noted that $\phi(\lambda)$ is a holomorphic function at any point $\lambda \in i\mathbf{R}$ and

$$(e^{\lambda\tau} - B)\, \hat{u}(\lambda) = e^{\lambda\tau}\phi(\lambda) + e^{\lambda\tau}\hat{f}(\lambda).$$

Since f is τ-periodic, if $\zeta \notin (2\pi/\tau)\mathbf{Z}$, $\hat{f}(\lambda)$ is holomorphic at $i\zeta$. On the other hand, for ζ such that $e^{i\zeta\tau} \in \rho(B)$, $R(e^{\lambda\tau}, B)$ is holomorphic with respect to λ at $i\zeta$. Thus, $\hat{u}(\lambda)$ has a holomorphic extension at $i\zeta$ provided that $e^{i\zeta\tau} \neq 1$, $e^{i\zeta\tau} \in \rho(B)$. Hence, by definition we have

$$e^{i\tau\,\mathrm{sp}(u)} \subset \{1\} \cup (\sigma(B) \cap S^1). \quad \blacksquare$$

In connection with Eq. (2) we consider the so-called *evolutionary semi-group* $(T^h)_{h \geqslant 0}$ in $\mathrm{AP}(\mathbf{X})$ which is defined by the formula

$$T^h v(t) = U(t, t-h)\, v(t-h), \qquad \forall v \in \mathrm{AP}(\mathbf{X}), \quad t \in \mathbf{R}, \quad h \geqslant 0. \tag{18}$$

LEMMA 4. *Let $(U(t,s))_{t \geqslant s}$ be a τ-periodic strongly continuous evolutionary process. Then its associated evolutionary semigroup $(T^h)_{h \geqslant 0}$ is strongly continuous in* $\mathrm{AP}(\mathbf{X})$. *Moreover, the infinitesimal generator of $(T^h)_{h \geqslant 0}$ is the operator $-L$ defined as follows: $u \in D(L)$ and $Lu = f$ if and only if $u, f \in \mathrm{AP}(\mathbf{X})$ and u is the solution to Eq.* (4).

Proof. For the proof see [N–M, Lemma 2]. $\blacksquare$

Proof of Theorem 1. First, by Lemmas 2 and 3, the spectrum of v of the system (12) is discrete. In fact, from the definition

$$\mathrm{sp}(v) \subset \mathrm{sp}(v_1) \cup \mathrm{sp}(v_2).$$

Since v_1 is τ-periodic, $\mathrm{sp}(v_1) \subset (2\pi/\tau)\,\mathbf{Z}$. Combining this and Lemma 3 we see that

$$\mathrm{sp}(v) \subset (2\pi/\tau)\,\mathbf{Z} \cup (\lambda_1 + 2\pi/\tau)\,\mathbf{Z} \cup \cdots (\lambda_N + 2\pi/\tau)\,\mathbf{Z}.$$

Hence, by Proposition 1 it is almost periodic. Moreover, by the Approximation Theorem [L–Z, Chap. 2] there is a sequence of trigonometric polynomials

$$v^{(n)}(t) = v_{n,0}(t) + \sum_{k=1}^{N} e^{i\lambda_k t} v_{n,k}(t), \tag{19}$$

where $v_{n,k}(t)$, $k = 0, ..., N$, are τ-periodic and continuous, which converges to v uniformly in $t \in \mathbf{R}$. (Note that N is independent of n.) By Lemma 1 and the Cauchy criterion for convergence it is seen that $v_{n,k}(t)$, $k = 0, ..., N$, converge to the τ-periodic continuous functions $v_k(t)$, $k = 0, 1, ..., N$, uniformly in t, respectively. Hence, we have proved that v has the form

$$v(t) = v_0(t) + \sum_{k=1}^{N} e^{i\lambda_k t} v_k(t), \tag{20}$$

where $v_k(t), k=0, 1, ..., N$, are τ-periodic continuous functions. Since $u(t)=Q^{-1}(t)\, v(t)$, where $Q^{-1}(t)$ is strongly continuous and τ-periodic, determined from (11), u is also of the form

$$u(t)=u_0(t)+\sum_{k=1}^{N} e^{i\lambda_k t}u_k(t). \tag{21}$$

We now show that u_0 is a τ-periodic mild solution to the inhomogeneous equation (1) and $u-u_0$ are τ-periodic mild solution to the homogeneous equation (2). To this end, equivalently, we will show that $Lu_0=f$, $L(u-u_0)=0$. In turn, by Lemma 4 this is equivalent to showing that

$$\lim_{h\to 0^+} \frac{T^h u_0-u_0}{h}=-f \tag{22}$$

$$\lim_{h\to 0^+} \frac{T^h(u-u_0)-(u-u_0)}{h}=0. \tag{23}$$

Set $w_h=(T^h u-u)/h$. Then applying Lemma 4, since u is an almost periodic solution of Eq. (4) $u\in D(L)$ and

$$\lim_{h\to 0^+} w_h=-Lu.$$

On the other hand, setting $w_k(t):=e^{i\lambda_k t}u_k(t)$ we have

$$\begin{aligned}\frac{w_k(t)-T^h w_k(t)}{h}&=e^{i\lambda_k t}\,\frac{u_k(t)-U(t,t-h)\,u_k(t-h)}{h}\\&\quad+e^{i\lambda_k t}\,\frac{1-e^{-i\lambda_k h}}{h}\,U(t,t-h)\,u_k(t-h), \qquad \forall t. \end{aligned}\tag{24}$$

Note that since u_k is τ-periodic and $(U(t,s))_{t\geqslant s}$ is τ-periodic and strongly continuous

$$v_{h,k}(t):=\frac{1-e^{-i\lambda_k h}}{h}\,U(t,t-h)\,u_k(t-h)$$

converges uniformly in t to $i\lambda_k u_k(t)$ as $h\to 0^+$. We have

$$w_h=w_{h,0}+\sum_{k=1}^{N} e^{i\lambda_k t}w_{h,k},$$

where

$$w_{h,0} = (T^h u_0 - u_0)/h; \; w_{h,k} = (T^h u_k - u_k)/h + v_{h,k};$$

$$v_{h,k}(t) = \frac{e^{-i\lambda_k h} - 1}{h} U(t, t-h)\, u_k(t-h), \forall t.$$

In view of Lemma 1, there exists $\lim_{h \to 0^+} w_{h,k}$, $\forall k$. Thus, the existence of $\lim_{h \to 0^+} v_{h,k}$ implies that of $(T^h u_k - u_k)/h$, $\forall k$. Consequently, we have $w_k, u_k \in D(L)$, $\forall k$ and

$$Lw_k(t) = e^{i\lambda_k t}(Lu_k(t) + i\lambda_k u_k(t)), \qquad \forall k = 1, ..., N. \tag{25}$$

Since Lu_k and u_k are τ-periodic we have

$$Lu(t) = Lu_0(t) + \sum_{k=1}^{N} e^{i\lambda_k t} v_k(t), \qquad \forall t \in \mathbf{R},$$

where $v_k(t) = i\lambda_k u_k(t) + Lu_k(t)$. Hence in view of Lemma 1, $Lu_0 = f$ and $L \sum_{k=1}^{N} e^{i\lambda_k t} v_k(t) = 0$. This completes the proof of the theorem. ∎

2.3. *When Does Boundedness Yield Uniform Continuity*?

It turns out that in many cases the uniform continuity follows from the boundedness of the solutions under consideration. Below we will discuss some particular cases frequently met in applications in which boundedness implies already uniform continuity.

DEFINITION 9. The τ-periodic strongly continuous evolutionary process $(U(t, s))_{t \geqslant s}$ is said to satisfy *condition C* if the (monodromy) operators $P(t) := U(t, t - \tau)$ is norm continuous with respect to t.

EXAMPLES. In the autonomous case, when $U(t, s) = T(t - s)$ for a compact C_0-semigroup $(T(t))_{t \geqslant 0}$, condition C is satisfied. Moreover, it can be checked easily that the perturbed equation

$$x(t) = T(t-s)\, x(s) + \int_s^t T(t-\xi)\, B(\xi)\, x(\xi)\, d\xi, \qquad \forall t \geqslant s, \tag{26}$$

where $t \mapsto B(t) \in L(\mathbf{X})$ is τ-periodic and continuous, determines a process satisfying condition C (see also Proposition 3 and Section 3 for parabolic equations which also satisfies condition C).

PROPOSITION 2. *Let the τ-periodic strongly continuous evolutionary process $(U(t, s))_{t \geqslant s}$ satisfy condition C and for every t the operator*

$P(t) := U(t, t-\tau)$ is compact. Then if u is a bounded mild solution to Eq. (4) *on the whole line, it is uniformly continuous.*

Proof. For the sake of simplicity we assume that $\tau = 1$. We consider the difference

$$u(t') - u(t) = U(t', t-1)\, u(t-1) - U(t, t-1)\, u(t-1) + \\ + \int_{t-1}^{t'} U(t', \xi)\, f(\xi)\, d\xi - \int_{t-1}^{t} U(t, \xi)\, f(\xi)\, d\xi. \tag{27}$$

We now show that

$$\lim_{|t'-t| \to 0} \| U(t', t-1) - U(t, t-1) \| = 0. \tag{28}$$

To this end, let us denote by B the unit ball of $\mathbf{X}$. Then the set $K = \{ P(t)\, x, t \in \mathbf{X}, x \in B \}$ is precompact. In fact, let $\varepsilon > 0$. Then by condition C and the τ-periodicity of $P(t)$ there are $0 < t_1 < \cdots < t_n < 1$ such that

$$\| P(t + t_i) - P(t + \xi) \| < \varepsilon/2, \qquad \forall \xi \in [t + t_{i-1}, t + t_i]. \tag{29}$$

Since for every t_i the operator $P(t_i)\, B$ is precompact, there exists $\{ x_1^i, x_2^i, ..., x_{k(i)}^i \} \subset B$ such that if $x \in B$ then

$$\| P(t_i)\, x_j^i - P(t_i)\, x \| < \varepsilon/2, \tag{30}$$

for some x_j^i. Now let $y = P(\eta)\, x$ for some $\eta \in \mathbf{R}$, $x \in B$. Then by the τ-periodicity of $P(t)$ we can assume $\eta \in [0, 1]$ and $\eta \in [t_{i-1}, t_i]$ for some i. From (29), (30) it follows that y is contained in the ball centered at $P(t_i)\, x_j^i$ with radius ε. This shows that K is precompact. Thus, (28) follows from

$$\lim_{t'-t \to 0^+} \sup_{x \in K} \| U(t'-t, 0)\, x - x \| = 0. \tag{31}$$

In turn, (31) follows from the precompactness of K and the strong continuity of the evolutionary process $(U(t, s))_{t \geqslant s}$. On the other hand, we have

$$\int_{t-1}^{t'} U(t', \xi)\, f(\xi)\, d\xi - \int_{t-1}^{t} U(t, \xi)\, f(\xi)\, d\xi \\ = \left[U(t', t) \int_{t-1}^{t} U(t, \xi)\, f(\xi)\, d\xi - \int_{t-1}^{t} U(t, \xi)\, f(\xi)\, d\xi \right] \\ + \int_{t}^{t'} U(t', \xi)\, f(\xi)\, d\xi. \tag{32}$$

By the 1-periodicity of $(U(t,s))_{t\geqslant s}$ and f it may be seen that $\int_{t-1}^{t} U(t,\xi) f(\xi)\,d\xi$ is 1-periodic with respect to t. Hence its range is precompact. As above, we see that

$$\lim_{t'-t\to 0^+}\left[U(t',t)\int_{t-1}^{t} U(t,\xi)\,f(\xi)\,d\xi-\int_{t-1}^{t} U(t,\xi)\,f(\xi)\,d\xi\right]=0$$

uniformly in t. This and the fact that

$$\lim_{t'-t\to 0^+}\int_{t}^{t'} U(t',\xi)\,f(\xi)\,d\xi=0,$$

imply that

$$\lim_{t'-t\to 0^+}\|u(t')-u(t)\|=0. \quad \blacksquare$$

Remarks. It is interesting to study the question of how many τ-periodic solutions Eq. (1) may have. This depends on the space of τ-periodic mild solutions of the homogeneous equation. A moment of reflection shows that if $v(\cdot)$ is a τ-periodic mild solution to the homogeneous equation (2), then $v(0)$ is a solution to the equation

$$y-Py=0,\ y\in\mathbf{X}$$

where P is the monodromy operator of the process $(U(t,s))_{t\geqslant s}$. Hence, if P is compact, then $v(0)$ belongs to a finite dimensional subspace of $\mathbf{X}$. That is the possible τ-periodic mild solutions to Eq. (1) forms a finite dimensional subspace. Note that in [N–M] we have shown that it is necessary and sufficient for Eq. (1) (without the compactness assumption) to have a unique τ-periodic solution that the unity belongs to the resolvent set of the monodromy operator P.

3. PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

In this section we will prove the analog of Theorem 1 for the functional differential equation

$$\frac{dx}{dt}=Ax+F(t)\,x_t+f(t),\qquad t\in\mathbf{R}, \tag{33}$$

where A is the generator of a compact C_0-semigroup, $F(t) \in L(C, \mathbf{X})$ is τ-periodic and continuous with respect to t, and f is continuous and τ-periodic. As usual, we denote $C := C([-r, 0], \mathbf{X})$, where $r > 0$ is a given real number. If $u: [s, s+\alpha) \mapsto \mathbf{X}$, $\alpha > r$ we denote $x_t(\theta) = x(t+\theta)$, $\theta \in [-r, 0]$, $t \in [s+r, s+\alpha-r)$.

We say that u is a *mild solution* to Eq. (33) on $[s, +\infty)$ if there is a $\phi \in C$ such that

$$u(t) = T(t-s)\,\phi(0) + \int_s^t T(t-\xi)[F(\xi)\,u_\xi + f(\xi)]\,d\xi, \quad \forall t \geqslant s. \tag{34}$$

u is said to be a *mild solution on the whole line* if instead of an arbitrary ϕ in Eq. (34) one takes u_s for every $s \in \mathbf{R}$. Under the above assumption, for every $s \in \mathbf{R}, \phi \in C$, there exists a unique mild solution $u(t)$ to Eq. (34) on $[0, \infty)$. And, by definition, the map taking $\phi \in C$ into $u_t \in C$ (denoted by $V(t, s)\,\phi$) in the case $f = 0$ is called *the solution operator* of the homogeneous equation associated with Eq. (34). From [T–W, Proposition 2.1] it may be noted that $(V(t, s))_{t \geqslant s}$ is a τ-periodic strongly continuous evolutionary process on C. For the sake of simplicity we assume that $\tau > r$. Thus the monodromy operator $V(\tau, 0)$ is compact [T–W, Proposition 2.4]. This will suggests us to generalize the result obtained in the previous section to Eq. (33).

Now suppose that we have the variation-of-constants formula (for the proof see [Wu, p. 116] for the autonomous case)

$$\begin{aligned} u(t) &= [V(t, s)\,\phi](0) + \int_s^t [V(t, \xi)\,X_0\,f(\xi)](0)\,d\xi, \\ u_0 &= \phi, \end{aligned} \tag{35}$$

for Eq. (33) where $X_0 : [-r, 0] \mapsto L(\mathbf{X})$ is given by $X_0(\theta) = 0 \;\; -r \leqslant \theta < 0$ and $X_0(0) = I$.

Hence from the τ-periodicity of the process $(V(t, s))_{t \geqslant s}$ and f we can write

$$u_t = V(t, t-\tau)\,u_{t-\tau} + g_t, \tag{36}$$

where g_t is τ-periodic and independent of u. Unfortunately, the validity of the variation-of-constants formula (35) is not clear (at least to us). However, we can show that the representation (36) holds true. In fact, we now show that

$$u(t+\theta)-[V(t,t-\tau)\,u_{t-\tau}](\theta)$$
$$=\int_{t-\tau}^{t+\theta} T(t+\theta-\xi)\,F(\xi)[u_\xi - V(\xi,t-\tau)\,u_{t-\tau}]\,d\xi +$$
$$+\int_{t-\tau}^{t+\theta} T(t+\theta-\xi)\,f(\xi)\,d\xi, \quad \theta\in[-r,0], \tag{37}$$

defines a τ-periodic function in C which is nothing but g_t. In turn, this follows immediately from the existence and uniqueness of solutions as shown in [T–W, Proposition 2.1] and the τ-periodicity of F, f. Hence in (36) g_t is a τ-periodic function which depends only on $(T(t))_{t\geqslant 0}$, $F(\cdot)$, $f(\cdot)$. Actually, g_t is the solution of (34) with initial value 0.

Thus we can apply the proof of the previous section to show that

$$u_t=u_0(t)+\sum_{k=1}^{M} e^{i\lambda_k t}u_k(t),$$

where u_j: $\mathbf{R}\mapsto C$ are τ-periodic and continuous for all $j=0, 1, ..., M$, and M is the number of basic oscillatory exponents of the process $(V(t,s))_{t\geqslant s}$. Hence

$$u(t)=u_t(0)=u_0(t)(0)+\sum_{k=1}^{M} e^{i\lambda_k t}u_k(t)(0), \qquad \forall t.$$

Finally we arrive at

$$u(t)=u^0(t)+\sum_{k=1}^{M} e^{i\lambda_k t}u^k(t), \qquad \forall t,$$

where $u^k(t):=u_k(t)(0)$ $\forall k=0, 1, ..., M$. We now check that u^0 is a τ-periodic mild solution to the inhomogeneous equation (33) and $\sum_{k=1}^{M} e^{i\lambda_k t}u^k$ is a quasi periodic mild solution to its homogeneous equation. In fact, to prove it we again use Lemma 4 to show that

$$Lu^0=\mathscr{F}u^0+f,$$

where $\mathscr{F}w(t)=F(t)\,w_t$. This can be done in the same manner as in the previous section.

THEOREM 2. *Let u be a bounded mild solution to Eq.* (33) *and $p\in\mathbf{N}$ such that $(p-1)\,\tau\leqslant r<p\tau$. Then u can be represented in the form*

$$u(t)=u^0(t)+\sum_{k=1}^{M} e^{i\lambda_k t}u^k(t), \forall t, \tag{41}$$

where u^0 is a $p\tau$-periodic mild solution to Eq. (33) *and $\sum_{k=1}^{M} e^{i\lambda_k t}u^k(t)$ a quasi periodic solution to its homogeneous equation.*

Proof. If u is uniformly continuous, then the proof can be done identically as in the previous section. The uniform continuity now follows from the boundedness as shown in Proposition 3 below. ∎

COROLLARY 2. *Eq.* (33) *has a τ-periodic mild solution if and only if it has a bounded mild solution. Moreover, if u is a bounded mild solution to Eq.* (33), *then the two-sided sequence*

$$a_k = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-(2ik\pi/\tau)\,\xi} u(\xi)\, d\xi,\ k\in \mathbf{Z}, \tag{39}$$

determines the Fourier coefficients of a τ-periodic mild solution to Eq. (33).

As in Proposition 2, for Eq. (33) we will show that the uniform continuity follows readily from the boundedness of the mild solution u. In fact we have

PROPOSITION 3. *Let $(T(t))_{t\geqslant 0}$ be a compact C_0-semigroup and let the maps $t\mapsto F(t)\in L(\mathbf{X})$, $t\mapsto f(t)\in \mathbf{X}$ be τ-periodic and continuous. Moreover, let u be a bounded mild solution to Eq.* (33) *on the whole line. Then it is uniformly continuous.*

Proof. We have to show that $\lim_{|t'-t|\to 0}\|u(t')-u(t)\|=0$. Without loss of generality we assume that $t'\geqslant t$. By the compactness of $(T(t))_{t\geqslant 0}$, the map $T(t)$ is norm continuous for $t>0$. Hence, we have

$$\begin{aligned}\lim_{t'-t\to 0^+}\|u(t')-u(t)\| &\leqslant \lim_{t'-t\to 0^+}\|T(t'-t+\tau)-T(\tau)\|\,\|u(t-\tau)\|\\ &\quad+\lim_{t'-t\to 0^+}\left\|\int_{t-\tau}^{t'} T(t'-\xi)[F(\xi)\,u_\xi+f(\xi)]\,d\xi\right.\\ &\quad\left.-\int_{t-\tau}^{t} T(t-\xi)[F(\xi)\,u_\xi+f(\xi)]\,d\xi\right\|\\ &\leqslant \lim_{t'-t\to 0^+}\left\|\int_{t-\tau}^{t'} T(t'-\xi)[F(\xi)\,u_\xi+f(\xi)]\,d\xi\right.\\ &\quad\left.-\int_{t-\tau}^{t} T(t-\xi)[F(\xi)\,u_\xi+f(\xi)]\,d\xi\right\|. \end{aligned}\tag{40}$$

On the other hand, by the exponential boundedness of the semigroup $(T(t))_{t\geqslant 0}$ and the boundedness of u, F, f,

$$
\begin{aligned}
\lim_{t'-t\to 0^+} & \left\| \int_{t-\tau}^{t'} T(t'-\xi)[F(\xi)\,u_\xi+f(\xi)]\,d\xi \right. \\
& \left. -\int_{t-\tau}^{t} T(t-\xi)[F(\xi)\,u_\xi+f(\xi)]\,d\xi \right\| \\
\leqslant & \lim_{t'-t\to 0^+} \left\| T(t'-t)\int_{t-\tau}^{t} T(t-\xi)[F(\xi)\,u_\xi+f(\xi)]\,d\xi \right. \\
& \left. -\int_{t-\tau}^{t} T(t-\xi)[F(\xi)\,u_\xi+f(\xi)]\,d\xi \right\| \\
& +\lim_{t'-t\to 0^+} \left\| \int_{t}^{t'} T(t'-\xi)[F(\xi)+f(\xi)]\,d\xi \right\| \\
\leqslant & \lim_{t'-t\to 0^+} \left\| T(t'-t)\int_{t-\tau}^{t} T(t-\xi)[F(\xi)\,u_\xi+f(\xi)]\,d\xi \right. \\
& \left. -\int_{t-\tau}^{t} T(t-\xi)[F(\xi)\,u_\xi+f(\xi)]\,d\xi \right\|.
\end{aligned}
\tag{41}
$$

As shown in [T–W, Lemma 2.5] the set $\int_c^\tau T(\eta)\,F(t+\eta)\,u_{t+\eta}\,d\eta$ is precompact for every fixed positive c. From this follows easily the precompactness of the set $\int_0^\tau T(\eta)\,F(t+\eta)\,u_{t+\eta}\,d\eta$ and then that of the set $\int_{t-\tau}^{t} T(t-\xi)[F(\xi)\,u_\xi+f(\xi)]\,d\xi$. This and the strong continuity of $(T(t))_{t\geqslant 0}$ imply that

$$
\begin{aligned}
\lim_{t'-t\to 0^+} & \left\| T(t'-t)\int_{t-\tau}^{t} T(t-\xi)[F(\xi)\,u_\xi+f(\xi)]\,d\xi \right. \\
& \left. -\int_{t-\tau}^{t} T(t-\xi)[F(\xi)\,u_\xi+f(\xi)]\,d\xi \right\| = 0.
\end{aligned}
\tag{42}
$$

Thus $\lim_{t'-t\to 0^+}\|u(t')-u(t)\|=0$. ∎

4. EXAMPLES

Let us consider the inhomogeneous parabolic equations of the form

$$
\frac{dx}{dt}+Ax=B(t)\,x+f(t), \qquad t\in\mathbf{R} \tag{43}
$$

and the homogeneous equations

$$\frac{dx}{dt} + Ax = B(t)\,x, \qquad t \in \mathbf{R}, \tag{44}$$

where A is sectorial in $\mathbf{X}$, $0 \leqslant \alpha < 1$, and $t \mapsto B(t) : \mathbf{R} \mapsto L(\mathbf{X}^\alpha, \mathbf{X})$ is τ-periodic and Hölder continuous (for more details see [He, p. 190]). Moreover, assume that A has compact resolvent. Then the homogeneous equation (44) generates an evolutionary process $(T(t, s))_{t \geqslant s}$ with the following property: $T(t, s)$ is compact for all $t \geqslant s$. Furthermore, if u is a bounded solution, then it is uniformly bounded (for this one can apply [He, Theorem 7.1.3]). Hence

COROLLARY 3. *If Eq.* (43) *has a bounded solution u on the whole line, then it has a τ-periodic solution u_0 whose Fourier coefficients are determined by the formula*

$$a_k = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-(2ik\pi/\tau)\,\xi} u(\xi)\, d\xi, \qquad k \in \mathbf{Z}. \tag{45}$$

Proof. As uniform continuity follows from boundedness and the notions of classical and mild solutions are the same, the proof is obvious from Theorem 1. ▮

ACKNOWLEDGMENT

The paper was written while N.V.M. was a fellow of the Japan Society for the Promotion of Science. The generous support of the Society is gratefully acknowledged. The authors thank S. Murakami for several remarks.

REFERENCES

[A] H. Amann, "Ordinary Differential Equations," de Gruyter Studies in Mathematics, Vol. 13, de Gruyter, Berlin, 1990.

[A–P] L. Amerio and G. Prouse, "Almost Periodic Functions and Functional Equations," Van Nostrand Reinhold, New York, 1971.

[A–S] W. Arendt and S. Schweiker, Discrete spectrum and almost periodicity, preprint.

[C–H] S. N. Chow and J. K. Hale, Strongly limit-compact maps, *Funkc. Ekvac.* **17** (1974), 31–38.

[D–M] D. Daners and P. K. Medina, "Abstract Evolution Equations, Periodic Problems and Applications," Pitman Research Notes in Mathematics, Vol. 279, Longman, New York, 1992.

[F] M. Farkas, "Periodic Motions," Applied Mathematical Sciences, Vol. 104, Springer-Verlag, New York, 1994.

[Ha] J. K. Hale, "Theory of Functional Differential Equations," Springer-Verlag, New York/Berlin, 1977.

[H–V] J. K. Hale and S. M. Verduyn-Lunel, "Introduction to Functional Differential Equations," Springer-Verlag, Berlin/New York, 1993.

[H–K] L. Hatvani and T. Kristin, On the existence of periodic solutions for linear inhomogeneous and quasilinear functional differential equations, *J. Differential Equation* **97** (1992), 1–15.

[He] D. Henry, "Geometric Theory of Semilinear Parabolic Equations," Lecture Notes in Mathematics, Springer-Verlag, Berlin/New York, 1981.

[H–M–N] Y. Hino, S. Murakami and T. Naito, "Functional Differential Equations with Infinite Delay," Lecture Notes in Mathematics, Vol. 1473, Springer-Verlag, Berlin/New York, 1991.

[H–M–Y] Y. Hino, S. Murakami and T. Yoshizawa, Existence of almost periodic solutions of some functional differential equations with infinite delay in a Banach space, *Tohoku Math. J.* **49** (1997), 133–147.

[La] C. Langenhop, Periodic and almost periodic solutions of Volterra integral differential equations with infinite memory, *J. Differential Equation* **58** (1985), 391–403.

[L–Z] B. M. Levitan and V. V. Zhikov, "Almost Periodic Functions and Differential Equations," Moscow Univ. Publ. House, 1978; English translation, Cambridge Univ. Press, Cambridge, UK, 1982.

[Li] J. Liu, Bounded and periodic solutions of differential equations in Banach space, *in* "Differential Equations and Computational Simulations, I, Mississippi State, MS, 1993," *Appl. Math. Comput.* **65**, Nos. 1–3 (1994), 141–150. [MR: 95i:34118]

[L–L–L] Y. Li, Z. Lin and Z. Li, A Massera type criterion for linear functional differential equations with advanced and delay, *J. Math. Anal. Appl.* **200** (1996), 715–725.

[Lu] A. Lunardi, "Analytic Semigroups and Optimal Regularity in Parabolic Problems," Birkhäuser, Basel, 1995.

[Mu1] S. Murakami, Linear periodic functional differential equations with infinite delay, *Funkc. Ekvac.* **29**, No. 3 (1986), 335–361.

[Mu2] S. Murakami, On the variation of constants formula for functional differential equations in Banach spaces, unpublished communication.

[N–M] T. Naito and Nguyen Van Minh, Evolution semigroups and spectral criteria for almost periodic solutions of periodic evolution equations, *J. Differential Equations* **152** (1999), 358–376.

[Paz] A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Applied Mathematical Sciences, Vol. 44, Springer-Verlag, Berlin/New York, 1983.

[Pr] J. Prüss, "Evolutionary Integral Equations and Applications," Birkhäuser, Basel, 1993.

[S–N] J. S. Shin and T. Naito, Semi-Fredholm operators and periodic solutions for linear functional differential equations, *J. Differential Equations* **153** (1999), 407–441.

[T–W] C. C. Travis and G. F. Webb, Existence and stability for partial functional differential equations, *Trans. Amer. Math. Soc.* **200** (1974), 395–418.

[W] J. Wu, "Theory and Applications of Partial Functional Differential Equations," Applied Mathematical Sciences, Vol. 119, Springer-Verlag, Berlin/New York, 1996.

[A] M. Yamaguchi, Almost periodic solutions of one dimensional wave equations with periodic coefficients, *J. Math. Kyoto Univ.* **29-3** (1989), 463–487.

[Yo] T. Yoshizawa, "Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions," Applied Mathematical Sciences, Vol. 14, Springer-Verlag, New York, 1975.

Printed in Belgium